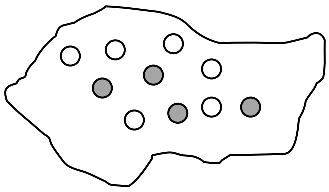
ELECTROMAGNETISM

GAUSS LAW OF ELECTROSTATICS:

It states that the total amount of electric flux through any closed surface is equal to $\frac{1}{\in}$ times the total number of charges enclosed by the surface.

○ $= +q_1, +q_2, +q_3, +q_4, +q_5, +q_6$

● $= -q_7, -q_8, -q_9, -q_{10}$



Mathematically,

$$\varphi_E = \oint_S \vec{E}\cdot d\vec{S} = \frac{Q_{NET}}{\in} \qquad (1)$$

In terms of displacement density, $\vec{D}$

$$\phi_E = \oint_S \vec{D}\cdot d\vec{S} = Q_{NET} \qquad \{\because \vec{D} = \in \vec{E}\} \qquad (2)$$

$$Q_{NET} = \left\{ \begin{array}{r} (q_1+q_2+q_3+q_4+q_5+q_6) \\ -(q_7+q_8+q_9+q_{10}) \end{array} \right\}$$

We know that the net charge in terms of charge density is given by,

$$Q_{NET} = \int_V \rho\, dV \qquad \text{where, } \rho = \text{Volume charge density} \qquad (3)$$

So, from equations (1), (2) and (3), we can write,

$$\oint_S \vec{E}\cdot d\vec{S} = \int_V \frac{\rho}{\in} dV \qquad (4)$$

and

$$\oint_S \vec{D}\cdot d\vec{S} = \int_V \rho\, dV \qquad (5)$$

For free space, $\in \to \in_0$

$$\oint_S \vec{E}\cdot d\vec{S} = \int_V \frac{\rho}{\in_0} dV \qquad (6)$$

Applying Gauss -Divergence theorem to equation (4) and (6), we can write

$$\int_V \left(\vec{\nabla}\cdot\vec{E}\right) dV = \int_V \frac{\rho}{\in} dV \qquad \text{For a medium} \qquad (7)$$

and,

$$\int_V \left(\vec{\nabla}\cdot\vec{E}\right) dV = \int_V \frac{\rho}{\in_0} dV \qquad \text{For free space} \qquad (8)$$

Equations (4), (5), (6), (7) and (8) represent integral form of Gauss's law in electrostatics.

From equation (5), we can write,

$$\oint_S \vec{D}\cdot d\vec{S} = \int_V \rho\, dV \qquad \Rightarrow \int_V \vec{\nabla}\cdot\vec{D} dV = \int_V \rho\, dV \qquad \text{(Applying G.D. Theorem)}$$

$$\Rightarrow \int_V \vec{\nabla}\cdot\vec{D} dV - \int_V \rho\, dV = 0 \qquad \Rightarrow \int_V (\vec{\nabla}\cdot\vec{D} - \rho) dV = 0$$

$$\Rightarrow \left(\vec{\nabla}\cdot\vec{D}\right)-\rho=0 \qquad (\because \text{ Volume is arbitrary})$$

$$\Rightarrow \vec{\nabla}\cdot\vec{D}=\rho \qquad (9)$$

From equation (9), we can write,

$$\vec{\nabla}\cdot\vec{D}=\rho \qquad \Rightarrow \vec{\nabla}\cdot\in\vec{E}=\rho \qquad (\because \vec{D}=\in\vec{E})$$

$$\Rightarrow \vec{\nabla}\cdot\vec{E}=\frac{\rho}{\in} \qquad (\because \in \text{ is a constant for medium}) \qquad (10)$$

For free space, $\in\to\in_0$ $\quad \vec{\nabla}\cdot\vec{E}=\frac{\rho}{\in_0}$ (11)

Equations (9), (10) and (11) represent differential form of Gauss's law in electrostatics.

Conclusion:

a. The total amount of charge enclosed by a surface may be positive or negative or zero. Accordingly, the flux may be outward, inward or zero.

b. Total charge enclosed is independent of the relative position or state of motion of charges as long as they are within the surface.

c. This law is applicable for both point charge distribution and continuous charge distribution.

d. The electric flux doesn't depend on the shape or size of the Gaussian surface.

Limitations:

a. This law is only helpful in calculating the magnitude of the electric field only.

b. This law can be used to calculate the electric flux only for those surfaces which has got the Gaussian surfaces in the form of simple geometry.

GAUSS'S LAW IN MAGNETISM

In case of Electrostatics, the line of force begins and terminates on positive and negative charge. However, the magnetic lines of force are closed on themselves without any beginning or end. So, magnetic poles occur in pairs. Isolated pole doesn't exist. So, the magnetic flux density over the closed surface is zero.

This is called Gauss's law in magnetism.

Mathematically, $\oint_S \vec{B}\cdot d\vec{S}=0$ (1)

Applying Gauss Divergence theorem, we can write $\int_V \vec{\nabla}\cdot\vec{B}dV=0$ (2)

Since the volume is arbitrary, $\vec{\nabla}\cdot\vec{B}=0$ (3)

$$\Rightarrow \vec{\nabla}\cdot\mu\vec{H}=0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{H}=0 \qquad (4)$$

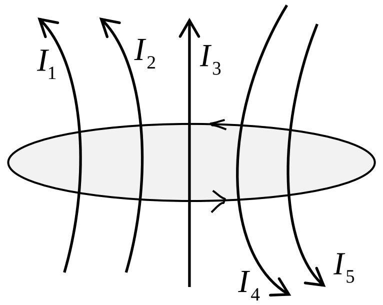
Equations (1) & (2) represent integral form of Gauss's law and equations (3) & (4) represent differential form of Gauss's law in magnetism.

AMPERE'S CIRCUITAL LAW:

This law states that the line integral of the magnetic field intensity $\vec{H}$ along any closed path is exactly equal to the total current enclosed by path.

The closed loop 'c' is called Amperian loop which can be of any shape as it encloses the currents.

The magnetic field can be determined by choosing a convenient shape of the Amperian loop.



$$\left[\oint_c \vec{H}\cdot d\vec{l} = I_{NET} = \left(I_1 + I_2 + I_3 - I_4 - I_5\right)\right]$$

Mathematically, $\oint_c \vec{H}\cdot d\vec{l} = I_{NET}$

We know that, $I = \iint_S \vec{J}\cdot d\vec{S}$, $\vec{J} =$ Current density

$$\therefore \oint_c \vec{H}\cdot d\vec{l} = \iint_S \vec{J}\cdot d\vec{S} \qquad (1)$$

$$\Rightarrow \iint_S \left(\vec{\nabla}\times\vec{H}\right)\cdot d\vec{S} = \iint_S \vec{J}\cdot d\vec{S} \qquad (2) \qquad \text{(Applying Stoke's Theorem)}$$

We know that, $\vec{B} = \mu\vec{H} \quad \Rightarrow \vec{H} = \dfrac{\vec{B}}{\mu}$; $\quad \mu =$ Permeability of the medium.

$\therefore$ Equations (1) and (2) can be written as,

$$\oint_c \vec{B}\cdot d\vec{l} = \mu\iint_S \vec{J}\cdot d\vec{S} \qquad (3)$$

$$\Rightarrow \iint_S \left(\vec{\nabla}\times\vec{B}\right)\cdot d\vec{S} = \mu\iint_S \vec{J}\cdot d\vec{S} \qquad (4) \qquad \text{(Applying Stoke's Theorem)}$$

For free space, $\mu \to \mu_0$

$$\therefore \oint_c \vec{B}\cdot d\vec{l} = \mu_0\iint_S \vec{J}\cdot d\vec{S} \qquad (5)$$

and $$\iint_s \left(\vec{\nabla}\times\vec{B}\right)\cdot d\vec{S} = \mu_0\iint_S \vec{J}\cdot d\vec{S} \qquad (6)$$

Equations (1), (2), (3), (4), (5) and (6) represent integral form of Ampere's circuital law.

Again from equation (2) we can write,

$$\iint_S \left(\vec{\nabla}\times\vec{H}\right)\cdot d\vec{S} = \iint_S \vec{J}\cdot d\vec{S} \qquad \Rightarrow \iint_S \left(\vec{\nabla}\times\vec{H}\right)\cdot d\vec{S} - \iint_S \vec{J}\cdot d\vec{S} = 0$$

$$\Rightarrow \iint_S \left\{\left(\vec{\nabla}\times\vec{H}\right) - \vec{J}\right\}\cdot d\vec{S} = 0$$

Since surface is arbitrary, we can write,

$$\left(\vec{\nabla}\times\vec{H}\right) - \vec{J} = 0 \qquad \Rightarrow \vec{\nabla}\times\vec{H} = \vec{J} \qquad (7)$$

$$\Rightarrow \vec{\nabla}\times\frac{\vec{B}}{\mu}=\vec{J} \qquad (\because \vec{B}=\mu\vec{H})$$

$$\Rightarrow \vec{\nabla}\times\vec{B}=\mu\vec{J} \qquad (8)$$

For free space, $\mu\to\mu_0$, $\quad \vec{\nabla}\times\vec{B}=\mu_0\vec{J} \qquad (9)$

Equation (7), (8) and (9) represents differential form of Ampere's circuital law.

The positive current is taken in the direction of advancing right-handed screw turn in which the direction is traversed. The law is analogous to Gauss's law in electrostatics and it is used to determine the magnetic field around the current carrying conductor for symmetrical configurations. To determine the field using this law the following two conditions must be satisfied.

a. At each point on the closed path, $\vec{H}$ is either tangential or normal to the curve.

b. $\vec{H}$ has the same value at all points on the path where it is tangential.

DISPLACEMENT CURRENT AND DISPLACEMENT CURRENT DENSITY:

When the field changes with time then the nature of current flowing through the circuit is known as displacement current. The displacement current can be well explained by considering the flow of current in a capacitive circuit.

In a capacitor, the current flow is possible if the voltage applied to it changes with time. In such case, the current through it is,

$$I=\frac{\partial Q}{\partial t}=C\frac{\partial V}{\partial t} \qquad (1) \qquad (\because Q=CV \text{ and } C = \text{capacitance of the capacitor})$$

Let us consider an a.c. circuit containing a capacitor of capacitance 'C', thickness 'd', area of cross section 'A' and permittivity '$\in$'. The conduction current in the wire is equal to the displacement current in the capacitor.

In a capacitive circuit, the displacement current is,

$$I_d=\frac{\in A}{d}\cdot\frac{\partial V}{\partial t} \qquad \Rightarrow I_d=\in A\frac{\partial\left(V/d\right)}{\partial t}$$

$$\Rightarrow I_d=\in A\frac{\partial\vec{E}}{\partial t} \qquad \left(\because E=\frac{V}{d}, \textit{taking the magnitude only}\right)$$

$$\Rightarrow I_d=A\frac{\partial}{\partial t}\left(\in\vec{E}\right) \qquad \Rightarrow I_d=A\frac{\partial D}{\partial t} \qquad (2) \qquad \sin ce, \vec{D}=\in\vec{E}$$

$$\Rightarrow \frac{I_d}{A}=\frac{\partial D}{\partial t} \qquad \Rightarrow \vec{J}_d=\frac{\partial\vec{D}}{\partial t} \qquad (3) \qquad (\vec{J}_d=\frac{I_d}{A}=\text{Displacement current density})$$

$$\Rightarrow \vec{J}_d=\in\frac{\partial\vec{E}}{\partial t} \qquad (4)$$

In Integral form, $\quad J_d=\in\frac{\partial}{\partial t}\oint_S\vec{E}\cdot d\vec{S}=\in\frac{\partial\varphi_E}{\partial t} \qquad (5)$

From equation (1) and (2), we can define the displacement current as the rate at which the displacement or flow of charge takes place from one electrode to the other.

From equation (3) and (4), the displacement current density can be defined as the rate of displacement of charges per unit area.

DIFFERENCE BETWEEN CONDUCTION CURRENT AND DISPLACEMENT CURRENT:

CONDUCTION CURRENT	DISPLACEMENT CURRENT
a. It is produced due to the actual flow of the charge carriers through the conductor.	**a.** This current is produced due to the time varying Electric field.
b. It obeys Ohm's law. $I_c = \frac{V}{R}$	**b.** It doesn't obey Ohm's law. $I_d = \in A\frac{\partial \vec{E}}{\partial t}$
c. The Conduction current lags behind the displacement current by a phase of $\pi/2$ when an alternating field is applied.	**c.** The displacement current leads the Conduction current by a phase of $\pi/2$ when an alternating field is applied.

MODIFIED AMPERE'S CIRCUITAL LAW / AMPERE-MAXWELL CIRCUITAL LAW:

Ampere's Circuital law is given by, $\boxed{\vec{\nabla} \times \vec{H} = \vec{J}}$ (1)

Ampere's circuital law couldn't able to explain the flow of current in circuits where the field changes with time. So, Maxwell modified the circuital law by introducing the displacement current arising due to a time varying field.

So, the law is known as Modified Ampere's circuital law or Ampere-Maxwell circuital law.

Mathematically, $\vec{\nabla} \times \vec{H} = \vec{J} + \vec{J}_d$

$$\Rightarrow \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad (2)$$

$$\Rightarrow \vec{\nabla} \times \vec{H} = \vec{J} + \in \frac{\partial \vec{E}}{\partial t} \qquad (3)$$

Equations (2) and (3) represent differential form of Ampere-Maxwell circuital law.

In integral form it can be written as,

$$\oiint_S \left(\vec{\nabla} \times \vec{H}\right) \cdot d\vec{S} = \oiint_S \left(\vec{J} + \in \frac{\partial \vec{E}}{\partial t}\right) \cdot d\vec{S} \qquad (4)$$

$$\Rightarrow \oint_c \vec{H} \cdot d\vec{l} = \oiint_S \left(\vec{J} + \in \frac{\partial \vec{E}}{\partial t}\right) \cdot d\vec{S} \qquad \text{(5) (Using Stoke's theorem)}$$

Equations (4) and (5) represent integral form of Ampere-Maxwell circuital law.

FARADAY'S LAW OF ELECTROMAGNETIC INDUCTION:

The law states that,

a. Whenever there is a change in the magnetic flux with time, an induced emf is produced in the circuit.

b. The induced emf remains only for the time for which the flux is actually changing.

c. The magnitude of the induced emf depends upon the rate at which the magnetic flux changes.

The direction of the induced emf is given by Lenz's law which states that the direction of the induced emf is such that it opposes the change for which it is produced.

Mathematically, $\varepsilon = -\frac{\partial \varphi_B}{\partial t}$ (1)

Where, φ_B = Magnetic flux linking through the circuit.

'-' ve is given as the emf opposes the change in flux that produces it.

We know that,

$$\varepsilon = \oint_c \vec{E} \cdot d\vec{l} \quad \text{and} \quad \varphi_B = \oint\!\!\oint_S \vec{B} \cdot d\vec{S}$$

So, equation (1) can be written as,

$$\oint_c \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \oint\!\!\oint_S \vec{B} \cdot d\vec{S} \qquad \Rightarrow \oint_c \vec{E} \cdot d\vec{l} = -\oint\!\!\oint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \qquad (2)$$

$$\Rightarrow \oint\!\!\oint_S \left(\vec{\nabla} \times \vec{E}\right) \cdot d\vec{S} = -\oint\!\!\oint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \qquad (3)$$

$$\Rightarrow \oint\!\!\oint_S \left(\vec{\nabla} \times \vec{E}\right) \cdot d\vec{S} + \oint\!\!\oint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} = 0 \qquad \Rightarrow \oint\!\!\oint_S \left\{\left(\vec{\nabla} \times \vec{E}\right) + \frac{\partial \vec{B}}{\partial t}\right\} \cdot d\vec{S} = 0 \quad (5)$$

$$\Rightarrow \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad (\because \text{The surface is arbitrary})$$

$$\Rightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (6)$$

$$\Rightarrow \vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \qquad (7) \qquad \left(\because \vec{B} = \mu \vec{H}\right)$$

Equations (6) and (7) represent differential form of Faraday's law and equations (2), (3) and (5) represent integral form of Faraday's law of Electromagnetic induction.

MAXWELL'S EQUATIONS IN ELECTROMAGNETISM:

In electrostatics, the differential equations valid are,

$$\vec{\nabla} \times \vec{E} = 0 \quad (1) \qquad \vec{\nabla} \cdot \vec{D} = \rho \quad (2)$$

Similarly, in magnetism, the differential equations valid are,

$$\vec{\nabla}\times\vec{H}=\vec{J} \qquad (3) \qquad\qquad \vec{\nabla}\cdot\vec{B}=0 \qquad (4)$$

In the static case, $\vec{E}$ and $\vec{D}$, & $\vec{B}$ and $\vec{H}$ they are not inter-related. But in case of time varying fields, $\vec{E}$ and $\vec{D}$ are related to $\vec{B}$ and $\vec{H}$. Due to this, equations (1) and (3) are needed to be modified.

Equation (1) is modified through Faraday's law of electro-magnetic induction and is given by,

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \qquad (5)$$

Equation (3) is modified through Maxwell by considering the fact that in a circuit in addition to the conduction current, another current exists which is due to the time variation of the electric field and is called displacement current. So equation (3) is modified and written as,

$$\vec{\nabla}\times\vec{H}=\vec{J}+\vec{J}_d \qquad \Rightarrow \boxed{\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}} \qquad (6)$$

This is called modified Ampere's circuital law or Ampere-Maxwell circuital law.

Maxwell combined equations (2), (4), (5) and (6) and gave a set of equations known as **Maxwell's equations** for electromagnetic waves or electromagnetism. The following equations are Maxwell's equations in electromagnetism in a medium with permittivity $\in$ and permeability μ.

$$\boxed{\vec{\nabla}\cdot\vec{D}=\rho} \qquad \boxed{\vec{\nabla}\cdot\vec{B}=0}$$

$$\boxed{\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}} \qquad \boxed{\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}}$$

Sl No.	DIFFERENTIAL FORM	INTEGRAL FORM	ORIGIN
1.	$\vec{\nabla}\cdot\vec{D}=\rho$	$\int_V \vec{\nabla}\cdot\vec{D}dV=\int_V \rho dV$ OR $\oint_S \vec{D}\cdot d\vec{S}=\int_V \rho dV$ (Applying Divergence theorem) OR $\oint_S \vec{E}\cdot d\vec{S}=\frac{1}{\in}\int_V \rho dV \ (\because \vec{D}=\in\vec{E})$	Gauss's law in electrostatics which states that the net amount of flux over a closed surface is equal to $1/\varepsilon$ times the net amount of charge enclosed by it.
2.	$\vec{\nabla}\cdot\vec{B}=0$	$\int_V \vec{\nabla}\cdot\vec{B}dV=0$ OR $\oint_S \vec{B}\cdot d\vec{S}=0$ (Applying divergence theorem)	Gauss's law in magnetism which states that magnetic monopole does not exist. Or Surface integral of magnetic induction over a closed surface is equal to zero.

3.	$\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$	$\oint\!\!\oint_S\left(\vec{\nabla}\times\vec{H}\right)\cdot dS=\oint\!\!\oint_S\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right)\cdot d\vec{S}$ OR $\oint_c\vec{H}\cdot d\vec{l}=\oint\!\!\oint_S\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right)\cdot d\vec{S}$ (Applying Stoke's theorem)	From Ampere's Maxwell Circuital law or modified Ampere's circuital law which states that the line integral of magnetic field intensity over a closed curve is equal to the sum of the conduction current density and the displacement current density.
4.	$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$ OR $\vec{\nabla}\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t}$	$\oint\!\!\oint_S\left(\vec{\nabla}\times\vec{E}\right)\cdot d\vec{S}=-\frac{\partial}{\partial t}\oint\!\!\oint_S\vec{B}\cdot d\vec{S}$ OR $\oint_c\vec{E}\cdot d\vec{l}=-\frac{\partial}{\partial t}\oint\!\!\oint_S\vec{B}\cdot d\vec{S}$ (Using Stoke's theorem)	From Faraday's law of electromagnetic induction which states that when the total flux linking through a circuit changes with time, an induced emf is produced in the circuit.

PHYSICAL SIGNIFICANCE OF MAXWELL'S EQUATIONS:

$$\vec{\nabla}\cdot\vec{D}=\rho \qquad (1) \qquad\qquad \vec{\nabla}\cdot\vec{B}=0 \qquad (2)$$

$$\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t} \qquad (3) \qquad\qquad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \qquad (4)$$

a. Maxwell's equations incorporate all the laws of electromagnetism.

b. These equations lead to the existence of electromagnetic waves.

c. These equations are consistent with special theory of relativity.

d. These equations are used to describe classical electromagnetic field as well as quantum electromagnetic field.

e. These equations provide a unified description of electric and magnetic phenomena which were treated independently.

f. Equations (1) and (2) represent the steady state equations because no time varying terms are there.

g. Equations (2) and (4) are known as constraint equations because these equations remain unaltered for any medium, in the absence of charge and current or presence of charge and current.

PROPERTIES OF ELECTROMAGNETIC WAVES:

1. Electromagnetic waves (EM) travel with speed of light.
2. Electromagnetic waves (EM) are transverse waves.
3. The ratio of electric and magnetic field in an electromagnetic wave is equal to the speed of light.
4. Electromagnetic waves carry both energy and momentum which can be delivered to a surface.

ELECTROMAGNETIC WAVE EQUATIONS FOR CONDUCTING MEDIA

From Maxwell's equations we know that the space variation of electric and magnetic field are related to time variation of magnetic and electric field components respectively. This interdependence gives rise to phenomenon of electromagnetic wave propagation.

So, the general wave equation can be derived by using Maxwell's equations for a conducting medium and it can be extended for finding the same for insulating medium and free space.

In case of conducting medium,

$\vec{D} = \in \vec{E}$ ($\in$ is the permitivity) $\quad$ $\vec{J} = g\,\vec{E}$ (g is the conductivity)

$\vec{B} = \mu \vec{H}$ (μ is the permeability) $\quad$ $\rho = 0$ (ρ is the volume charge density)

From Maxwell's equations, we can write

(a) $\vec{\nabla}\cdot\vec{D} = \rho \quad \Rightarrow \quad \vec{\nabla}\cdot \in \vec{E} = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{E} = 0 -----------(1) \quad (\because \in \neq 0)$

(b) $\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot \mu\vec{H} = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{H} = 0 -----------(2) \quad (\because \mu \neq 0)$

(c) $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

$$\Rightarrow \vec{\nabla}\times\left(\vec{\nabla}\times\vec{H}\right) = \vec{\nabla}\times\vec{J} + \vec{\nabla}\times\frac{\partial \vec{D}}{\partial t} \quad \text{(Taking curl of both sides)}$$

$$\Rightarrow \vec{\nabla}\left(\vec{\nabla}\cdot\vec{H}\right) - \nabla^2\vec{H} = \vec{\nabla}\times g\vec{E} + \vec{\nabla}\times\frac{\partial\left(\in\vec{E}\right)}{\partial t} \quad \text{.(using vector identity, } \vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \vec{\nabla}\left(\vec{\nabla}\cdot A\right) - \nabla^2 A\text{)}$$

$$\Rightarrow \quad -\nabla^2\vec{H} = g\left(\vec{\nabla}\times\vec{E}\right) + \in\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{E}\right) \quad (\because g \text{ and } \in \text{ are constants for a medium \& } \vec{\nabla}\cdot\vec{H} = 0)$$

$$\Rightarrow \quad -\nabla^2\vec{H} = g\left(-\frac{\partial\vec{B}}{\partial t}\right) + \in\frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right) \quad \text{(From Maxwell's equation } \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\text{)}$$

$$\Rightarrow \quad -\nabla^2\vec{H} = -g\mu\frac{\partial\vec{H}}{\partial t} - \in\mu\frac{\partial^2\vec{H}}{\partial t^2} \quad (\because \vec{B} = \mu\vec{H})$$

$$\Rightarrow \quad \nabla^2\vec{H} = g\mu\frac{\partial\vec{H}}{\partial t} + \in\mu\frac{\partial^2\vec{H}}{\partial t^2}$$

$$\Rightarrow \quad \boxed{\nabla^2\vec{H} - \in\mu\frac{\partial^2\vec{H}}{\partial t^2} - g\mu\frac{\partial\vec{H}}{\partial t} = 0} \qquad (3)$$

(d) $$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Taking curl of both sides,

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = -\vec{\nabla}\times\frac{\partial\vec{B}}{\partial t}$$

$$\Rightarrow \quad \vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\mu\vec{H}\right)$$ (using vector identity, $\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}\left(\nabla\cdot A\right) - \nabla^2 A$)

$$\Rightarrow \quad -\nabla^2\vec{E} = -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\mu\vec{H}\right)$$ ($\because \vec{B} = \mu H$ & $\vec{\nabla}\cdot\vec{E} = 0$)

$$\Rightarrow \quad \nabla^2\vec{E} = \mu\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{H}\right)$$ $\because \mu$ is a constant

$$\Rightarrow \quad \nabla^2\vec{E} = \mu\frac{\partial}{\partial t}\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right)$$ (From Ampere Maxwell’s equation)

$$\Rightarrow \quad \nabla^2\vec{E} = \mu\frac{\partial}{\partial t}\left(g\vec{E}\right) + \mu\frac{\partial^2}{\partial t^2}\left(\in\vec{E}\right)$$ ($\because J = gE$ & $D = \in E$)

$$\Rightarrow \quad \nabla^2\vec{E} = \mu g\frac{\partial}{\partial t}\vec{E} + \in\mu\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\Rightarrow \quad \boxed{\nabla^2\vec{E} - \in\mu\frac{\partial^2\vec{E}}{\partial t^2} - \mu g\frac{\partial}{\partial t}\vec{E} = 0} \qquad (4)$$

Equations (3) and (4) represent electromagnetic wave equations in terms of magnetic field intensity $\vec{H}$ and electric field intensity $\vec{E}$ in a homogeneous conducting medium.

The third term in equations (3) and (4) is a dissipating term which is equivalent to the damping term in the equation for damped harmonic motion. Due to this term, the electromagnetic waves damped very quickly in a conducting medium and can’t propagate.

For dielectric or non conducting medium,

$$g = 0 \quad \Rightarrow \quad \vec{J} = g\vec{E} = 0$$

So, the electromagnetic wave equation can be written as

$$\boxed{\nabla^2\vec{H} - \in\mu\frac{\partial^2\vec{H}}{\partial t^2} = 0} \qquad (5)$$

and

$$\boxed{\nabla^2 \vec{E} - \in \mu \frac{\partial^2 \vec{E}}{\partial t^2} = 0} \qquad (6)$$

Equation (5) and (6) represents electromagnetic wave equations in case of dielectric or non-conducting medium.

For free space, $\in \rightarrow \in_0$ *and* $\mu \rightarrow \mu_0$

$$\therefore \quad \boxed{\nabla^2 \vec{H} - \in_0 \mu_0 \frac{\partial^2 \vec{H}}{\partial t^2} = 0} \text{ ------------------------------------- } (7)$$

$$\boxed{\nabla^2 \vec{E} - \in_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0} \text{ --------------------------------------- } (8)$$

ELECTROMAGNETIC WAVES TRAVEL WITH SPEED OF LIGHT:

The electromagnetic wave equations for free space is given by

$$\left.\begin{aligned} \nabla^2 \vec{H} - \in_0 \mu_0 \frac{\partial^2 \vec{H}}{\partial t^2} = 0 \\ \nabla^2 \vec{E} - \in_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \end{aligned}\right\} \qquad (1)$$

The classical wave equation is given by,

$$\nabla^2 \varphi - \frac{1}{v^2} \frac{\partial^2 \varphi}{\partial t^2} = 0 \qquad (2)$$

v is the velocity with which the wave moves.

Comparing equations (1) and (2) we can write,

$$\frac{1}{v^2} = \in_0 \mu_0 \quad \Rightarrow \quad v^2 = \frac{1}{\in_0 \mu_0}$$

$$\Rightarrow \quad v = \frac{1}{\sqrt{\in_0 \mu_0}}$$

$$\Rightarrow \quad v \cong 3 \times 10^8 \, m/\sec = \text{velocity of light}$$

$\Rightarrow$ Electromagnetic waves travel with the speed of light and light is an electromagnetic wave.

General solution of the electromagnetic wave equation.

The general solution for the electromagnetic wave equation is given by,

$$\boxed{\vec{E} = \hat{e} E_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}} \qquad (1)$$

and, $\boxed{\vec{B} = \hat{b}\, B_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}}$ (2)

or $\boxed{\vec{H} = \hat{b}\, H_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}}$ (3)

Where, $\hat{b}$ & $\hat{e}$ represents the direction of magnetic field and electric field respectively.

E_0 & B_0 represents the amplitude of electric and magnetic field respectively.

$\vec{k}$ is the propagation vector that represents the direction of propagation of the waves.

$\vec{r} \longrightarrow$ Position vector t $\longrightarrow$ time.

ELECTROMAGNETIC WAVES ARE TRANSVERSE WAVES:

From Maxwell's equation for a charge free conducting medium, we can write

$$\vec{\nabla}\cdot\vec{D} = 0$$

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{E} = 0 \qquad (\because \vec{D} = \in \vec{E} \quad and \quad \in \neq 0)$$

$$\Rightarrow \quad \vec{\nabla}\cdot\left\{\hat{e}\, E_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right\} = 0 \qquad \text{(Using the solution for } \vec{E}\text{)}$$

$$\Rightarrow \quad \left(\hat{e}\cdot\vec{k}\right) iE_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} = 0$$

Since $i = \sqrt{-1} \neq 0$ *and* $E_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \neq 0$ being the solution

So, $\boxed{\hat{e}\cdot\vec{k} = 0}$

$\Rightarrow$ Electric waves are transverse in nature.

Again, from Maxwell's equation we know,

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\Rightarrow \quad \vec{\nabla}\cdot\left\{\hat{b}\, B_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right\} = 0 \qquad \text{(Using the solution for } \vec{B}\text{)}$$

$$\Rightarrow \quad i\, B_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}\left(\vec{k}\cdot\hat{b}\right) = 0$$

Since $i = \sqrt{-1} \neq 0$ *and* $B_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \neq 0$ being the solution,

So, $\vec{k}\cdot\hat{b} = 0 \quad \Rightarrow \hat{b}\cdot\vec{k} = 0$

$\Rightarrow$ Magnetic waves are transverse in nature.

Now, from Maxwell's equation we can write,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

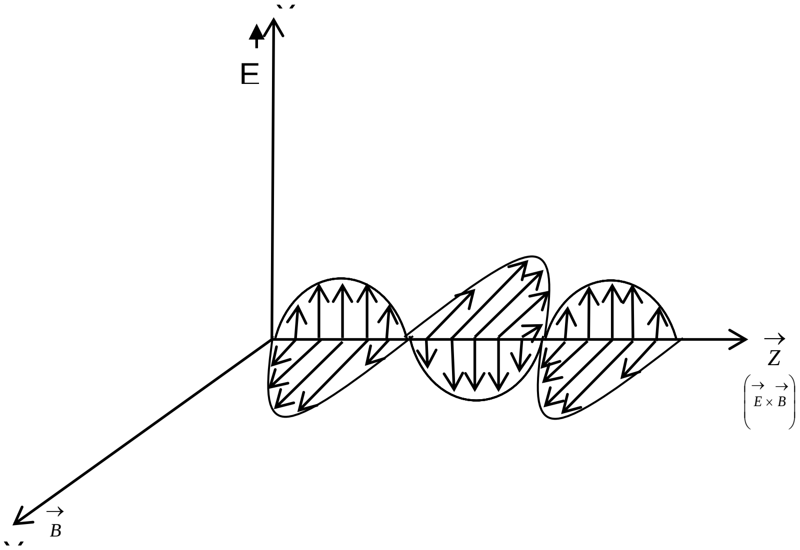
$$\Rightarrow \qquad \vec{\nabla} \times \left[\hat{e} E_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right] = -\frac{\partial}{\partial t} \left[\hat{b} B_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right]$$

$$\Rightarrow \qquad E_0 \left(\vec{k} \times \hat{e} \right) = \omega B_0 \hat{b}$$

$\Rightarrow$ $\quad \vec{k}, \hat{e}\, and\, \hat{b}$ are mutually perpendicular to each other.

$\Rightarrow$ $\quad$ Electromagnetic waves are transverse in nature.

If the electric field is taken along x – direction, magnetic field along y – direction then the electromagnetic waves will propagate along z – direction.



ELECTROMAGNETIC ENERGY DENSITY

The electric energy density or electric energy per unit volume is,

$$\boxed{u_E = \frac{1}{2}\vec{D}.\vec{E} = \frac{1}{2} \in \vec{E}.\vec{E} = \frac{1}{2} \in E^2} \text{ ------------------------ (1)}$$

where $\vec{D} = \in \vec{E}$

The magnetic energy density is,

$$\boxed{u_B = \frac{1}{2}\vec{B}.\vec{H} = \frac{1}{2}\mu \vec{H}.\vec{H} = \frac{1}{2}\mu H^2} \text{ ---------------- (2)}$$

where $\vec{B} = \mu \vec{H}$

So, the electromagnetic energy density is

$$u_{EM} = u_E + u_B$$

$$\boxed{u_{EM} = \frac{1}{2}\left(\in E^2 + \mu H^2 \right) = \frac{1}{2}\left(\vec{D}.\vec{E} + \vec{B}.\vec{H} \right)} \text{ ------------------ (3)}$$

The electromagnetic energy can be obtained by taking the volume integral of the above equation

POYNTING VECTOR:

A vector that describes the rate of energy transfer per unit area in EM wave is known as poynting vector which was named after J..H. Poynting. It is denoted by $\vec{S}$. It measures the flow of electromagnetic energy per unit time per unit area normal to the direction of wave propagation.

In terms of electric and magnetic field vectors, $\vec{S}$ can be written as,

$$\boxed{\vec{S} = \vec{E} \times \vec{H}}$$

The direction of Poynting vector is perpendicular to both electric and magnetic fields and is directed along the direction of propagation of the wave.

Its unit is watt/m^2.

ENERGY FLOW IN AN ELECTROMAGNETIC WAVE AND POYNTING THEOREM :

An important aspect of wave propagation is the flow of power through space. Traveling waves carry energy with them and at any instant there is flow of power per unit area at the surface.

The theorem that describes the rate of energy transfer or flow of power through the surface is known as Poynting theorem.

It states that, the net power flowing out of a given volume v is equal to the time rate of decrease in the energy stored within the volume and the ohmic power dissipated within the volume or conduction losses.

Mathematically,

$$\boxed{\int_v \vec{\nabla}.\vec{S}.dV = -\frac{\partial}{\partial t}\int_v \frac{1}{2}\left(\in E^2 + \mu H^2\right)dV - \int_v \vec{J}.\vec{E}.dV} \text{------------------------} (1)$$

$$\boxed{\oint_S \vec{S}.d\vec{S} = -\frac{\partial}{\partial t}\int_v \frac{1}{2}\left(\in E^2 + \mu H^2\right)dV - \int_v \vec{J}.\vec{E}.dV} \text{------------------------} (2)$$

Where, $\frac{1}{2} \in E^2 + \frac{1}{2}\mu H^2$ = Electric energy + Magnetic energy

$\vec{J}.\vec{E}$ = Ohmic power.

RELATIVE MAGNITUDES OF ELECTRIC AND MAGNETIC FIELDS:

From the transverse nature of electromagnetic waves, we know that

$$\vec{k} \times \hat{e} = \frac{\omega B_0}{E_0}\hat{b} \quad \Rightarrow \quad k = \frac{\omega B_0}{E_0} \qquad \text{(Taking the magnitude only)}$$

$$\Rightarrow \quad k = \frac{2\pi\nu B_0}{E_0} \quad \Rightarrow \quad k = \frac{\frac{2\pi}{\lambda} c B_0}{E_0} \quad \Rightarrow \quad k = \frac{k c B_0}{E_0}$$

$$\Rightarrow \quad \frac{E_0}{B_0} = c \quad \Rightarrow \quad \frac{E_0}{\mu H_0} = c \quad \Rightarrow \quad \frac{E_0}{H_0} = \mu c$$

$$\Rightarrow \quad \frac{E_0}{H_0} = \frac{\mu}{\sqrt{\epsilon_0 \mu_0}}$$

For vacuum, $\mu \longrightarrow \mu_0, \therefore \quad \frac{E_0}{H_0} = \frac{\mu}{\sqrt{\epsilon_0 \mu_0}} \quad \Rightarrow \quad \frac{E_0}{H_0} = \sqrt{\frac{\mu_0}{\epsilon_0}} = z_0$ (Impedance of the vacuum).

In case of electromagnetic wave the electric and magnetic waves are always in phase. Since magnitude and direction of electromagnetic waves are related, so any one of them can be used to describe the Electromagnetic wave. Generally, the electric field is chosen for most of the cases.

POYNTING VECTOR AND INTENSITY OF EM WAVES :

In Electromagnetic field the electric and magnetic fields are time varying fields. So, the Poynting vector also changes with time.

We know, $\vec{S} = \vec{E} \times \vec{H}$

$$\Rightarrow \quad S = EH = \frac{EB}{\mu} \qquad (1) \qquad \text{[Taking the magnitude only]}$$

Since in case of em waves the electric waves and magnetic waves are in phase, so the ratio of their instantaneous values is same as the ratio of their maximum values.

We know, $\frac{E_0}{B_0} = c \quad \Rightarrow \quad \frac{E}{B} = c \Rightarrow \quad B = \frac{E}{c}$

So, $S = \frac{EB}{\mu} \quad \Rightarrow \quad S = \frac{E^2}{\mu c}$

If, $E = E_0 \sin \omega t$, the time average value of Poynting vector is,

$$\langle S \rangle = \frac{\langle E_0 \sin^2 \omega t \rangle}{c\mu}$$

$$\Rightarrow \quad \langle S \rangle = \frac{1}{2}\frac{E_0^2}{c\mu} \quad \Rightarrow \quad \langle S \rangle = \frac{E_{rms}^2}{c\mu} \quad \left[\because \quad \langle \sin^2 \omega t \rangle = \frac{1}{2} \text{ and } E_{rms} = \frac{E_0}{\sqrt{2}}\right]$$

So, the intensity of EM wave is nothing but the time average of Poynting vector.

$$\therefore \quad \boxed{I = \langle S \rangle = \frac{E_{rms}^2}{c\mu}}$$